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$SL(2, C)$ Gravitational Conserved Current and Noether's Theorem

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Noether's theorem is applied to Hilbert's Lagrangian written as a functional of spinorial variables. The associated $SL(2, C)$ conserved current is obtained, and its expression for the Tolman metric is given explicitly.

1. INTRODUCTION

It is well known that Hilbert's Lagrangian density

$$\mathscr{L}_H = (-g)^{1/2}R - 2\kappa\mathscr{L}_M \tag{1}$$

when considered as a functional of the metric and the affine connections, leads through a Palatini-type variational principle to the Einstein field equations (Palatini, 1919; Misner et al., 1973)

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{2}$$

The field equations are obtained by varying the components of the metric, whereas the variation of the affine connections leads to its definition in terms of the geometrical metric,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2}g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}) \tag{3}$$

It was shown by Carmeli and Kaye (Carmeli and Kaye, 1978; Carmeli, 1982) that an $SL(2, C)$ Palatini variational formalism can be performed on Hilbert's Lagrangian written as a functional of spinorial variables,

$$\mathscr{L}_H = \mathscr{L}_0(\sigma^{\mu}_{ab'}, B_{\mu a}{}^{b}, B_{\mu,\sigma a}{}^{b}) - 2\kappa\mathscr{L}_M \tag{4}$$

(Greek letters denote, as usual, tensor indices with values 0 to 3. Lower case Latin indices denote dyad components of spinors in a local spinor

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frame and take the values 0, 1. Primed indices denote complex conjugation of spinors.) $\sigma^{\mu}_{ab'}$ and $B_{\mu a}{}^{b}$ are 2×2 complex matrices the elements of which are 4-vectors. The first, $\sigma^{\mu}_{ab'}$, is defined by the requirement that (Penrose, 1960)

$$\sigma^{\mu}_{ab'}\sigma^{\nu ab'}=g^{\mu\nu} \tag{5}$$

where $g^{\mu\nu}$ is the geometrical metric. The second, $B_{\mu a}{}^{b}$, constitutes the vector potential in the $SL(2, C)$ gauge theory of gravitation (Carmeli, 1982). It is traceless, that is, it belongs to the internal space of the $SL(2, C)$ group.

The explicit form of the gravitational part of this Lagrangian density is

$$\mathscr{L}_0=-2(-g)^{1/2}\sigma^{\mu\nu}{}_{a}{}^{b}F_{\mu\nu b}{}^{a} \tag{6}$$

where

$$\sigma^{\mu\nu}{}_{a}{}^{b}=\sigma^{[\mu}_{ac'}\sigma^{\nu]bc'} \tag{7}$$

(The square brackets denote antisymmetrization of the enclosed indices.) Hence $\sigma^{\mu\nu}{}_{a}{}^{b}$ is a metric skew-symmetric spinorial tensor. It is easily seen from its definition that it belongs to the internal space of the $SL(2, C)$ group, namely, its components are 2×2 traceless complex matrices. The field variable $F_{\mu\nu a}{}^{b}$, appearing in equation (6), is the gauge gravitational field tensor given by the elements of the matrix

$$F_{\mu\nu}=B_{\mu,\nu}-B_{\nu,\mu}+[B_{\mu}, B_{\nu}] \tag{8}$$

It is worthwhile noticing that although equations (1) and (4) express the same Lagrangian, they are structured differently. While the former is obtained from the product of two symmetric tensors, the latter is obtained from two antisymmetric tensors.

Varying with respect to the components of the $\sigma^{\mu}_{ac'}$, one obtains the Einstein gravitational field equations in dyad notation. In turn, the Lagrange equations which result from the variation with respect to the matrix elements of the vector potential, $B_{\mu a}{}^{b}$, give the definition of this potential in terms of the $\sigma^{\mu}_{ac'}$ along with their directional derivatives (Carmeli and Kaye, 1978).

In this paper we apply Noether's theorem to this Lagrangian density in order to get a metric conserved current resulting from the $SL(2, C)$ symmetry of the gravitational field. In Section 2 we briefly review the conservation laws related to the $SL(2, C)$ symmetry that arise from a quadratic Lagrangian density. In Section 3 we apply Noether's theorem to the Carmeli–Kaye Lagrangian and obtain a conserved current. An application is subsequently made in Section 4 where the conserved current is calculated explicitly for the Tolman metric. The last section is devoted to the concluding remarks.

2. *SL*(2, *C*) GRAVITATIONAL CONSERVED CURRENTS OBTAINED FROM QUADRATIC LAGRANGIAN

The well-known decomposition of the Riemann curvature tensor to its three irreducible tensorial components induces a division of the gauge field (Carmeli, 1982)

$$F^{\mu\nu}{}_a{}^b = F^{W\mu\nu}{}_a{}^b + F^{S\mu\nu}{}_a{}^b + F^{T\mu\nu}{}_a{}^b = F^{W\mu\nu}{}_a{}^b + F^{R\mu}{}_a{}^b \tag{9}$$

Carmeli (1976) demonstrated the existence of a conserved vector current obtained from this division, which shows a remarkable similarity to Yang–Mill's (1954) current. In matrix notation it is given by

$$J^{W\alpha} = \kappa \hat{J}^\alpha + \tfrac{1}{2}\varepsilon^{\alpha\beta\mu\nu}[B_\beta, F^W_{\mu\nu}] = \tfrac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F^W_{\mu\nu,\beta} \tag{10}$$

where

$$\hat{J}^\alpha = \frac{1}{2\kappa}\,\varepsilon^{\alpha\beta\mu\nu}(F^R_{\mu\nu,\beta} - [B_\beta, F^R_{\mu\nu}]) \tag{11}$$

Malin (1977), in turn, showed the existence of a conserved current that is given by

$$J^\alpha = \tfrac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu,\beta} = \tfrac{1}{2}\varepsilon^{\alpha\beta\mu\nu}[B_\beta, F_{\mu\nu}] \tag{12}$$

In addition to the above-mentioned induced division, the gauge field tensor decomposes into two tensors which are, respectively, the rotor and the commutator of the gauge potential (Nissani, 1983, 1984a):

$$L_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} \tag{13}$$

$$K_{\mu\nu} = [B_\mu, B_\nu] \tag{14}$$

$$F_{\mu\nu} = L_{\mu\nu} + K_{\mu\nu} \tag{15}$$

From the tensor $L_{\mu\nu}$ we obtain a tensorial continuity equation (Nissani, 1983, 1984a),

$$*\mathscr{L}^{\mu\nu}{}_{,\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}L_{\alpha\beta,\nu} = 0 \tag{16}$$

Applying Noether's theorem to the quadratic Lagrangian density of Carmeli (1976) for the $SL(2,C)$ gauge theory of gravitation we obtain, from the $SL(2,C)$ symmetry, the following conserved current (Carmeli and Nissani, 1982):

$$J^\mu_g = \tfrac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\mathrm{Tr}(F_{\alpha\beta}\, g)_{,\nu} \tag{17}$$

where g is an arbitrary traceless 2×2 complex matrix. If we now take $g = g_i$ $(i = 1, 2, 3)$ as a global basis of the internal space of the symmetry group, that is, $g_{,\mu}=0$, we obtain from equation (17)

$$j_i^\mu = \mathrm{Tr}(J^\mu g_i) \tag{18}$$

where J^μ is Malin's conserved current.

From the decompositions (9) and (15) of the gauge field we obtain two decompositions of this current to the conserved currents (Nissani, 1983)

$$j_i^\mu = \mathrm{Tr}(J^{W\mu} g_i) + \mathrm{Tr}(J^{R\mu} g_i) \tag{19}$$

$$j_g^\mu = j_L^\mu + j_K^\mu \tag{20}$$

where $J^{W\mu}$ is Carmeli's current, $J^{R\mu}$ is a conserved current related to the energy-momentum tensor (Nissani, 1983, 1984a),

$$J_L^\mu = \mathrm{Tr}(\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} L_{\alpha\beta}\, g)_{,\nu} \tag{21}$$

and

$$J_K^\mu = \mathrm{Tr}(\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} K_{\alpha\beta}\, g)_{,\nu} = \mathrm{Tr}(J^\mu g) + \mathrm{Tr}(\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} K_{\alpha\beta}\, g_{,\nu}) \tag{22}$$

Furthermore, one can show (Nissani, 1984b) that Carmeli's and Malin's currents belong to a set of six conserved currents of the form

$$\begin{aligned} J_{\mathrm{Ei}}^{X\mu} &= (-g)^{1/2} f_i^{X\mu\nu}{}_{;\nu} \qquad (X = W, S, T) \\ J_{\mathrm{Mi}}^{X\mu} &= (-g)^{1/2} {}^*f_i^{X\mu\nu}{}_{;\nu} \end{aligned} \tag{23}$$

Here, use is made of the notation

$$f_i^{X\mu\nu} = F^{X\mu\nu}{}_a{}^b g_{ib}{}^a \tag{24}$$

where $g_{ib}{}^a$ $(i = 1, 2, 3)$ is a local basis of the internal space of the $SL(2,C)$ group.

3. METRIC CONSERVED CURRENT OBTAINED FROM HILBERT'S LAGRANGIAN

In the previous section we reviewed the conserved currents related to the $SL(2,C)$ symmetry of the gravitational field through a quadratic Lagrangian density. In this section we apply Noether's theorem to Hilbert's Lagrangian density, written as a functional of spinorial variables, equations (4) and (6), to obtain a conserved current, which can be called a metric conserved current.

Using Anderson's notation (Anderson, 1967) the Noether theorem can be stated, in our case, as follows:

(i) Let G be a Lie and symmetry group of a theory which is sustained by a Lagrangian density $\mathscr{L}$;

(ii) The Lagrangian density depends only on the variables Y_A and their first derivatives $Y_{A,\mu}$, that is, $\mathscr{L}=\mathscr{L}(Y_A, Y_{A,\mu})$;

(iii) The Lagrangian density is invariant under the group transformations; and

(iv) The group does not operate upon the coordinates (internal group); then Noether's theorem acquires the very simple form

$$\partial_\mu\left(\sum_A \frac{\partial\mathscr{L}}{\partial Y_{A,\mu}} W_{gA}\right)=0 \tag{25}$$

where $\varepsilon W_{gA}=\delta_g Y_A$ are the variations due to the infinitesimal element of the group, $(I+\varepsilon g)$. Therefore, the expression

$$\sum_A \frac{\partial\mathscr{L}}{\partial Y_{A,\mu}} W_{gA}=j_g^\mu \tag{26}$$

defines a conserved current. If the symmetry group possesses n infinitesimal generators g_i $(i=1,\ldots,n)$, one obtains n conserved quantities,

$$j_i^\mu=\sum_A \frac{\partial\mathscr{L}}{\partial Y_{,A\mu}} W_{g_iA} \tag{27}$$

Since the $B_{\mu a}{}^b{}_{,\nu}$ are the only quantities with derivatives that appear in $\mathscr{L}_0$, and assuming that the matter Lagrangian is independent of spinorial derivative variables, we get

$$j_{\sigma i}^\mu=\frac{1}{4}\frac{\partial\mathscr{L}_0}{\partial B_{\nu a}{}^b{}_{,\mu}}\delta_{g_i}B_{\nu a}{}^b \tag{28}$$

where the factor 1/4 was introduced for the sake of convenience. The variation $\delta_{g_i}B_{\mu a}{}^b$ of the vector potential $B_{\mu a}{}^b$ [caused by an infinitesimal $SL(2,C)$ transformation, $(1+\varepsilon g_i)$, of the dyadic local frame] is obtained from its law of transformation,

$$B'_\mu=S^{-1}B_\mu S-S^{-1}S_{,\mu} \tag{29}$$

Here we obtain

$$j_{\sigma i}^\mu=(-g)^{1/2}\sigma^{\mu\nu}{}_a{}^b([B_\nu, g_i]-g_{i,\nu})_b{}^a \tag{30}$$

Using now the identity

$$\mathrm{Tr}(A[B, C])=\mathrm{Tr}([A, B]C)$$

we get

$$j_{\sigma i}^\mu=(-g)^{1/2}([\sigma^{\mu\nu}, B_\nu]_a{}^b g_{ib}{}^a-\sigma^{\mu\nu}{}_a{}^b g_{i,\nu b}{}^a) \tag{31}$$

But from the definition of the gauge potential (Carmeli, 1982),

$$\zeta^A_{a;\nu} = B_{\nu a}{}^b \zeta_b{}^A \tag{32}$$

we obtain

$$\sigma^{\mu\nu}{}_a{}^b{}_{;\rho} = [B_\rho, \sigma^{\mu\nu}]_a{}^b \tag{33}$$

Therefore, equation (31) becomes

$$j^\mu_{\sigma i} = -(-g)^{1/2} (\sigma^{\mu\nu}{}_a{}^b g_{ib}{}^a)_{;\nu} = -[(-g)^{1/2} \sigma^{\mu\nu}_i]_{,\nu} \tag{34}$$

where use is made of the notation

$$\sigma^{\mu\nu}_i = \sigma^{\mu\nu}{}_a{}^b g_{ib}{}^a \tag{35}$$

The $j^\mu_{\sigma i}$ are three conserved vector densities constructed from metric elements.

4. AN APPLICATION TO THE TOLMAN METRIC

In the last section we derived a conserved vector density, $j^\mu_{\sigma i}$, associated with the $SL(2,C)$ internal symmetry of the gravitational field. Here we give its expression for the Tolman metric describing a spherically dust cloud (Tolman, 1934).

The energy-momentum tensor of a dust cloud is given by

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{36}$$

where ρ is the mass density and $u^\mu = dx^\mu / ds$ is the 4-velocity. Tolman's solution of Einstein's field equations in spherically symmetric comoving coordinates, (t, r, θ, φ), is given by (Tolman, 1934)

$$ds^2 = dt^2 - \frac{R'^2}{1+f(r)} dr^2 - R^2 \, d\theta^2 - R^2 \sin^2 \theta \, d\varphi^2 \tag{37}$$

Here R is a function of t and r satisfying the condition

$$R' = \frac{\partial R}{\partial r} > 0 \tag{38}$$

whereas

$$\dot{R}^2 = \left(\frac{\partial R}{\partial t}\right)^2 = f(r) + \frac{F(r)}{R} \tag{39}$$

with

$$\frac{\partial F(r)}{\partial r} = \kappa \rho R^2 R' \tag{40}$$

$\dot{R}$ expresses the radial velocity of the dust particles, and for $f(r)=0$ we have

$$\dot{R}^2 = F(r)/R \tag{41}$$

that is, a parabolic motion.

In order to find the expression of the new current for the Tolman metric we take Pauli matrices and the unit matrix divided by $\sqrt{2}$ as the $\sigma^{(\mu)}_{ab'}$ vector in the local Minkowskian frame,

$$\sigma^{(0)}_{ab'} = \frac{I}{\sqrt{2}}, \qquad \sigma^{(k)}_{ab'} = \frac{\tau_k}{\sqrt{2}} \tag{42}$$

where I is the 2×2 unit matrix and τ_k are Pauli's matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We also take the following basis for the interior space of the $SL(2,C)$ group,

$$g_{ka}{}^{b} = \frac{\tau_k}{\sqrt{2}} \tag{43}$$

where $k=1,2,3$. Then we obtain for the skew-symmetric tensor $\sigma_i^{(\mu)(\nu)}$, in the local Minkowskian frame, the following:

$$\sigma_1^{(\mu)(\nu)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad \sigma_2^{(\mu)(\nu)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\sigma_3^{(\mu)(\nu)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

Now taking the matrix

$$\Lambda_{(\nu)}{}^{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & [R'/(1+f)^{1/2}]^{-1} & 0 & 0 \\ 0 & 0 & R^{-1} & 0 \\ 0 & 0 & 0 & (R\sin\theta)^{-1} \end{pmatrix} \tag{45}$$

which fulfills the condition

$$\Lambda_{(\mu)}{}^{\rho}\Lambda_{(\nu)}{}^{\sigma}\eta^{(\mu)(\nu)} = g^{\rho\sigma} \tag{46}$$

as the matrix of the local coordinate transformation from the local Minkowskian frame to the comoving coordinates, we obtain for $\sigma_1^{\mu\nu}$,

$$\sigma_1^{\mu\nu}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -(1+f)^{1/2}/R' & 0 & 0 \\ (1+f)^{1/2}/R' & 0 & 0 & 0 \\ 0 & 0 & 0 & i/R^2\sin\theta \\ 0 & 0 & -i/R^2\sin\theta & 0 \end{pmatrix} \tag{47}$$

Therefore the conserved vector density is given by

$$j_{\sigma 1}^{\mu}=-[(-g)^{1/2}\sigma_1^{\mu\nu}]_{,\nu}=(\sqrt{2}RR'\sin\theta, -\sqrt{2}R\dot{R}\sin\theta, 0, 0) \tag{48}$$

and the invariant conserved quantity, that is, the "charge" density measured in its rest frame, is

$$q=\left(\frac{1}{-g}j_{\sigma 1}^{\mu}j_{\sigma 1\mu}\right)^{1/2}=\frac{\sqrt{2}}{R}(1+f-\dot{R}^2)^{1/2} \tag{49}$$

The coordinate velocity of this "charge" is given by

$$\frac{dr}{dt}=\frac{j_{\sigma 1}^{1}}{j_{\sigma 1}^{0}}=-\frac{\dot{R}}{R'} \tag{50}$$

Hence its measured velocity with respect to the dust particle is

$$v=\frac{dl}{dt}=(-g_{11})^{1/2}\frac{dr}{dt}=-\frac{\dot{R}}{(1+f)^{1/2}} \tag{51}$$

and q takes the form

$$q=[2(1+f)]^{1/2}\frac{(1-v^2)^{1/2}}{R} \tag{52}$$

5. CONCLUDING REMARKS

We have seen how the $SL(2,C)$ group analysis of Einstein's equations leads to a conserved current. This is done, as is usually the case when one looks for conserved quantities, by applying Noether's theorem to the Lagrangian of the physical system (which, in our case, was the Carmeli-Kaye Lagrangian). Finally, the conserved current was given explicitly for the Tolman metric.

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